16.3 Wilcoxon two-sample test (Wilcoxon rank-sum test).

Example: The nicotine content of two brands of cigarettes, measured in milligrams was found to be as follows

Brand A:	2.1	4.0	6.3	5.4	4.8	3.7	6.1	3.3		
Brand B:	4.1	0.6	3.1	2.5	4.0	6.2	1.6	2.2	1.9	5.4

$H_0: \tilde{\mu}_A = \tilde{\mu}_B$ $\quad H_1: \tilde{\mu}_A \neq \tilde{\mu}_B$. | Distributions are assumed continuous and equal except for location.

Sample from brand A has $n_1 = 8$

Sample from brand B has $n_2 = 10$

Arrangement in ascending order.

	0.6	1.6	1.9	2.1	2.2	2.5	3.1	3.3	3.7	4.0	4.0	4.1	4.8	5.4	5.4	6.1	6.2	6.3
Rank	1	2	3	4	5	6	7	8	9	10.5	10.5	12	13	14.5	14.5	16	17	18
	B	B	B	A	B	B	B	A	A	B	A	B	A	B	A	A	B	A

Let W_1 be the sum of the ranks for the smallest sample and let W_2 — " — " — for the largest sample.

$$W_1 + W_2 = \frac{(n_1+n_2)(n_1+n_2+1)}{2}$$

Let us define $U_1 = W_1 - \frac{n_1(n_1+1)}{2}$ and $U_2 = W_2 - \frac{n_2(n_2+1)}{2}$

and let U be $\min(U_1, U_2)$

$$\text{Max } W_1 = \frac{(n_1+n_2)(n_1+n_2+1)}{2} - \frac{n_2(n_2+1)}{2}$$

$$= \frac{n_1(n_1+1) + n_1 n_2 + n_2(n_2+1) + n_1 n_2 - n_2(n_2+1)}{2} = n_1 n_2 + \frac{n_1(n_1+1)}{2}$$

Similarly: Max $W_2 = m_1 m_2 + \frac{m_2(m_2+1)}{2}$

and we get that U_1 and U_2 both take values in the interval $[0, m_1 m_2]$ and their distribution is symmetric

Relevant hypothesis

H_0	H_1	Test statistic
$\tilde{\mu}_1 = \tilde{\mu}_2$	$\tilde{\mu}_1 < \tilde{\mu}_2$	U_1
	$\tilde{\mu}_1 > \tilde{\mu}_2$	U_2
	$\tilde{\mu}_1 \neq \tilde{\mu}_2$	U

Here $U_{1.obs} = W_{1\,obs} - \frac{8.9}{2} = 93 - 36 = 57$

$U_{2\,obs} = W_{2\,obs} - \frac{10(11)}{2} = 78 - 55 = 23$.

Critical value from Table A.17 is 17 if $\alpha = 0.05$

$\Rightarrow$ there is no reason to reject H_0 on a 5% level.

Calculations by hand is possible but tedious.

For the ranks of U_2 there are $\binom{18}{10}$ possible ways to place 10 Bs among the total of 18 As and Bs

Have to find $P(U \leq u) = \sum_{u=1}^{23} P(U = u)$

and $P(U = u) = \frac{\#\text{ ways to place 10 Bs such that } W_2 - 55 = u}{\binom{18}{10}}$

When both n_1 and n_2 exceed 8, we can use that

U_1 and U_2 both are $\approx N\left(\frac{n_1 n_2}{2}, \frac{n_1 n_2 (n_1 + n_2 + 1)}{12}\right)$.

16.4 Kruskal-Wallis test

Test for $k > 2$ samples

Example. Battery life time for 3 types of mobiles measured in hours.

A	B	C
10.9 12.1 10.3 10.6 11.2	11.5 11.4 12.2 11.8 11.5 11.2 10.8	12.4 12.8 11.6 12.5 12.3 12.6
$n_1 = 5$	$n_2 = 7$	$n_3 = 6$

$n = 18$

$H_0: \tilde{\mu}_A = \tilde{\mu}_B = \tilde{\mu}_C$ $\qquad H_1$: At least one is different from the others.

Arranging the observations in ascending order we get:

	A	A	B	A	A	B	B	B	B	C	B	A	B	C	C	C	C	C
	10.3	10.6	10.8	10.9	11.2	11.2	11.4	11.5	11.5	11.6	11.8	12.1	12.2	12.3	12.4	12.5	12.6	12.8
							7	8.5	8.5	10	11	12	13	14	15	16	17	18
Rank	1	2	3	4	5.5	5.5												

$R_1 = 1 + 2 + 4 + 5.5 + 12 = 24.5$ $\qquad \bar{R}_1 = \frac{R_1}{n_1} \approx 4.9$

$R_2 = 3 + 5.5 + 7 + 8.5 + 8.5 + 11 + 13 = 56.5$ $\qquad \bar{R}_2 = \frac{R_2}{n_2} = 8.7$

$R_3 = 10 + 14 + 15 + 16 + 17 + 18 = 90$ $\qquad \bar{R}_3 = \frac{R_3}{n_3} = 15$

Test statistics:
$$H = \frac{12}{n(n+1)} \sum_{j=1}^{k} n_j \left(\bar{R}_j - \frac{n+1}{2}\right)^2$$

$$= \frac{12}{n(n+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(n+1) \approx \chi^2(k-1)$$

$$H_{obs} = \frac{12}{18 \cdot 19} \left(\frac{(24.5)^2}{5} + \frac{(56.5)^2}{7} + \frac{90^2}{6} \right) - 3 \cdot 19 = 10.58$$

$\chi^2_{0.05}(2) = 5.99 \Rightarrow$ We reject H_0. At least one is different.